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The Blockage Problem in a Traffic Model with Variable Speed

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Introduction

Traffic flow problems have become common social problems. Theoretical study of traffic flow has been mainly based on the methods of fluid dynamics,infact many theoretical results on the traffic flow in linear roads were obtained by using fluid-dynamical ideas. However this holds just for cases in which the car concentration is very low. A different approach is needed in situations of very congested traffic, as for example in a traffic jam: one of the simplest models describing the traffic flow on a linear road is the rule-184 elementally CA, according to the naming scheme by Wolfram. In contrast to the fluid dynamical modeling, cars are treated as distinguishable particles in CA models, the roads are expressed by discrete lattices and the system evolves in discrete time step. Owing to this space-time discretization, the CA models are easy to be tracted by computer simulations.

Hence it can be noticed that the CA model of traffic flow is closely related to the one-dimensional Totally Asymmetric Simple-Exclusion Process(TASEP).

The Totally Asymmetric Simple Exclusion Process (TASEP) is one of the more popular example of discrete particle system driven by a Markov irreversible dynamics. The system can be defined, in finite space, on a discrete segment $\Lambda = 1, 2, ..., 2L$, or on a discrete circle, imposing periodic boundary condition to the segment. A configuration $\sigma \in \{0, 1\}^\Lambda$ can be viewed as a set of particles living in Λ. According to this map, $\sigma_i = 1$ means that the ith site is occupied by a particle, whereas if $\sigma_i = 0$ then the ith site is a hole, i. e. it is an empty site. TASEP may be formulated either as a serial or a parallel dynamics. The *serial* TASEP chooses an occupied site i uniformly at random; if the site $i + 1$ is a hole, then the selected particle jumps to the $(i+1)$th site; conversely the $(i+1)$th site is occupied by another particle, then the configuration does not change and a new iteration starts over. The *parallel* TASEP selects instead all the particles having an unoccupied right-neighbouring site, but those actually advancing to the empty site are chosen according to a binomial rule, i. e. each particle actually advances with an independent probability p. Both serial and parallel dynamics are clearly irreversible.

Despite its simplicity, this model has many interesting features. On a finite circle the stationary measure is uniform because its transition matrix is doubly Markov, while on the finite segment the stationary state depends on the boundary probability to enter (say on the left) and to exit (on the right) from the system.

An interesting quantity to measure is the *current*, defined as the probability, under stationary conditions, to have a particle in a given site, with an empty site on its right. This quantity does not depend on the point where it is computed and it is important because it measures the tendency of the system to exhibit congestion, i.e. the tendency to form long sequences of clustered particles that are not free to move. The current can be exactly computed in the models mentioned above, and considering, for instance, the model defined on the circle, it is possible to see that the current depends only on the number of particles in the system, and it is maximum (and equal to 1/4 in the limit $L \to \infty$) when the system is half-filled, i.e. when there are L particles in the circle $\Lambda = \{1, 2, ..., 2L\}$.

In this thesis, we study if the effect of a blockage in the dynamics of the model has local effects, as in the case of reversible system far from critical points, or global effects. In the case of the TASEP this question has been investigated imposing the so-called blockage: in a defined point (say, without loss of generality, in the point 2L of the circle) the probability to jump to the empty site 1 if the particle in $2L$ is selected is $1 - \varepsilon$ with $\varepsilon > 0$. This blockage mimics the effect of the bottleneck caused by the road construction, the tunnel,and so on in the real world.

To check if the effect on the system is global or not, we must evaluate the current: if a blockage in a single point affects the value of the current then the effects of that blockage are obviously global. For a long time it has been unclear whether the presence of a blockage of intensity ε had global effects for all $\varepsilon > 0$. It is conjectured that the current decreases, for small ε, with a nonanalytic dependence on ε. Numerical evaluation of the current suggested the existence of a critical value $\varepsilon_c > 0$ such that the current does not change for $\varepsilon < \varepsilon_c$. Only very recently it has been proved that for $\Lambda = Z$ and continuous time, it is $\varepsilon_c = 0$. However, the conjecture about the non-analyticity of the current around $\varepsilon = 0$ still remains unproved. In the end we will study the parallel TASEP dynamics, where at each step each particle followed by an empty site has a finite probability p to jump. We call this parallel dynamics PCA-TASEP. We show that this model has similar features with respect to the standard TASEP; in particular, considering the blockage problem, we see numerically that for $p < 1$ the behaviour of the current is very similar to the standard TASEP case, because for small ε the current suggests a non-analytic dependence on ε around $\varepsilon = 0$. On the other hand, for $p = 1$, the system is exactly solvable and it can be proved that the current is analytic as a function of $0 \leq \varepsilon \leq 1$. In the last section of this thesis, we simulate the behaviour of the PCA-TASEP under variable speed condition, where the variable speed is translated as the possibility for the particle to jump a second time during the same iteration of the system with a probability q. Since this is a new situation, we do not have mathematical tools yet to describe the Markov chain, however we provide some qualitative explanation for the results after having observed how the model evolves during each iteration.

Chapter 1

PCA-TASEP

1.1 Definition of PCA-TASEP

A *PCA-TASEP* is a Markov chain defined on a discrete circle, i.e. on the set $\Lambda = 1, 2, ..., 2L$ with periodic boundary conditions, whose configurations σ are a points in the space $\{0, 1\}^{\Lambda}$. Denoting by σ_i the local configuration of σ in the point i, we will say that there is a particle in site i if $\sigma_i = 1$, otherwise, for $\sigma_i = 0$, we have a hole. Of course the particle in the ith state will be free to move if the following state presents a hole, or, in mathematical expression, if $\sigma_{i+1} = 0$. Hence, considering the dynamics of the system described in the introduction, it can be easily seen that to move a particle means to substitute the values $\sigma_i = 1$ and $\sigma_{i+1} = 0$ with $\sigma_i = 0$ and $\sigma_{i+1} = 1$.

Given a configuration σ we call $m(\sigma) = \sum_{i=1}^{2L} \sigma_i$ the number of particles living in Λ and, due to the dynamics of the model, we know that $m(\sigma)$ is conserved during the evolution of the system. So now we can define the state space of the Markov chain: it is a subset of $\{0, 1\}^{\Lambda}$ where the number of ones is always equal to $m(\sigma)$. Since the analysis will consider the half-filled PCA-TASEP ($m(\sigma) = L$), we will assume that $m(\sigma) \leq L$.

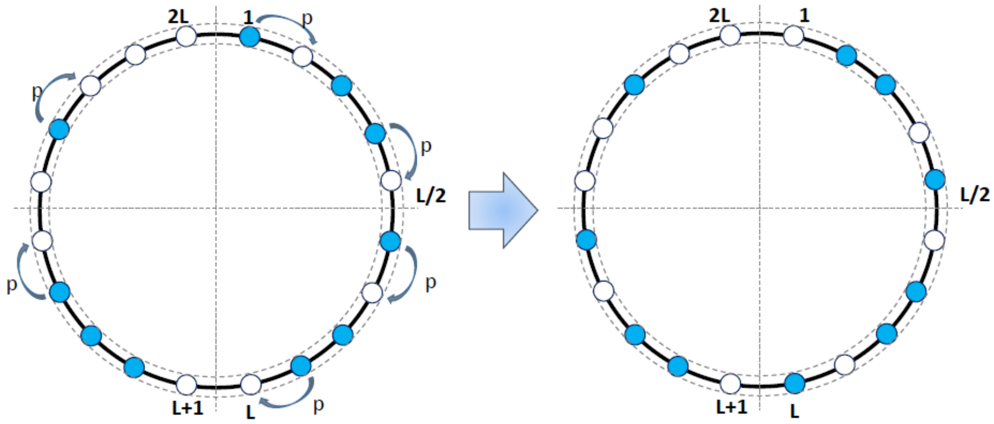


Figure 1.1: Representation of the PCA-TASEP model and a possible transition from one general configuration to another, obeying rule-184

1.2 Transition probabilities of the PCA-TASEP

Now we need to determine the transition probabilities to describe the Markov chain: a transition from the configuration σ to the configuration τ is weighed according to the following rule:

$$w(\sigma,\tau) = \begin{cases} w^n & \text{if } \tau \text{ can be reached by moving } n \text{ particles in } \sigma \\ 0 & \text{otherwise} \end{cases} \tag{1.1}$$

where $w > 0$ is a positive parameter measuring the tendency to move of each particle. Since a probability must be less than one, we have to divide by a normalization factor $w(\sigma)$ defined as:

$$w(\sigma) = \sum_{\tau} w(\sigma,\tau) = \sum_{k=0}^{l(\sigma)} \left(\begin{array}{c} l(\sigma) \\ k \end{array} \right) w^k = (1+w)^{l(\sigma)}$$

where l(σ) is the number of free particle of a configuration, such that $l\,(\sigma) \leq m\,(\sigma)$.

The term $w(\sigma)$ is the sum of the weights of all transition starting from σ, however this sum can be restricted to just all the possible transitions starting from σ, which depends obviously on the value of $l(\sigma)$.Then we can have transitions which require to move just one particular particle among all $l(\sigma)$, or two particles, up to transitions which requires all particles to move; so, if we consider the general case of k particles needed, the number of possible transitions is a combination of order $l(\sigma)$ and class k. Multiplying by w^k, we get the weight of all possible transitions obtained by moving k particles. Next step is to add the weights for all possible values of k.This explains the presence of the binomial coefficient in the equation and the indexes of the sum, which can be solved immediately, simply by recognizing the Newton's Binomial Formula.

Hence the transition probabilities are:

$$P(\sigma, \tau) = \frac{w(\sigma, \tau)}{w(\sigma)} = \frac{w^n}{(1+w)^{l(\sigma)}}$$

For small values of w, for example $O(1/L)$, the parallel TASEP has the same dynamics of the serial one, while for finite values of w, the dynamics is truly parallel; infact each free particle advances to the empty neighbouring site with independent probability p, computed as $p = w/1 + w$. Note that as $w \to \infty$, $p = 1$ and this implies that all the particles simultaneously move at each step(*rule-184 automata*).

1.3 Stationary Distribution of PCA-TASEP

Definition 1.3.1. *A stationary distribution π is a vector whose entries are the probabilities for the system to be in a configuration σ as the time tends to ∞, so when the Markov chain reaches an equilibrium condition. A particular property of π is the following:*

Given P, the Transition Matrix associated to a Markov chain, we have that $\pi P = \pi$ or equivalently $\sum_\sigma \pi(\sigma) P(\sigma, \tau) = \pi(\tau)$

Even if the PCA-TASEP is irreversible, it is possible to find the stationary distribution π of the chain by exploiting the *global balance principle*, or the *dynamical reversibility*, which states:

$$if\ the\ condition: \sum_\tau w(\sigma, \tau) = \sum_{\tau'} w(\tau', \sigma) is\ satisfied,\ then$$

$$the\ stationary\ distribution\ \pi(\sigma)\ is:\ \pi(\sigma) = w(\sigma)/W$$

where

$$W = \sum_{\sigma} w(\sigma).$$

The equation holds because the final configurations τ can be mapped one-to-one onto the initial configurations τ' at the right hand side in such a way that $w(\sigma, \tau) = w(\tau', \sigma)$.

Therefore, to prove that the result for $\pi(\sigma)$ is correct, we have to verify if the property decribed in definition 1.3.1 is respected.

Proof. Our candidate for the stationary distribution was $\pi(\sigma) = w(\sigma)/W$ with $W = \sum_{\sigma} w(\sigma)$. Substituting in the sum,

$$\sum_{\sigma} \pi(\sigma)P(\sigma, \tau) = \sum_{\sigma} \frac{w(\sigma)}{W} \frac{w(\sigma, \tau)}{w(\sigma)} = \sum_{\sigma} \frac{w(\sigma, \tau)}{W}$$

Now we apply the *global balance principle*

$$\sum_{\sigma} \frac{w(\sigma, \tau)}{W} = \sum_{\sigma} \frac{w(\tau, \sigma)}{W} = \frac{w(\tau)}{W} = \pi(\tau)$$

□

We have proved that the stationary measure is

$$\pi(\sigma) = \frac{(1 + w)^{l(\sigma)}}{W}$$

.

1.4 The Current for the Half-Filled PCA-TASEP

In this section, we will find an analytical equation to describe the current in the *half-filled* case, in which $m(\sigma) = m = L$ since the number of particle is conserved during the process.

Definition 1.4.1. *The value of the current J for the irreversible Markov chain, defined by the transition probabilities found in 1.2, is:*

$$J = \lim_{\Lambda \to \infty} \pi(\sigma_i = 1, \sigma_{i+1} = 0)$$

where π is the stationary distribution studied in 1.3.

However, since the event $\{\sigma_i = 1, \sigma_{i+1} = 0\}$ does not depend on the site i but only on the number of particle m, we can express the current J as the *expectation* $\mathbb{E}$ of $l(\sigma)$ with respect the stationary distribution π, normalized by a factor $2L$:

$$J = \lim_{\Lambda\to\infty} \pi(\sigma_i = 1, \sigma_{i+1} = 0) = \lim_{L\to\infty} \frac{\mathbb{E}_\pi\left[l(\sigma)\right]}{2L}$$

Before continuing with our calculations, we need to consider two very important properties, one of the expectation operator $\mathbb{E}$ and one of the stationary distribution π.

Property 1.4.2. *Given a sequence of random variables of a stochastic process, i.e. a Markov chain, X_n, with $n \in \mathbb{N}$, which assume values in some state space S, we know that under stationary conditions*

$$\lim_{n\to\infty} P(X_n = \sigma) = \pi(\sigma) \text{ for any } \sigma \in S$$

Property 1.4.3. *Given a random variable X with density $p(x)$, which is equivalent to $P(X = x)$, and a function f of the random variable X, we know that also $f(X)$ is a random variable, whose expectation $\mathbb{E}(f(X))$ is given by*

$$\mathbb{E}(f(X)) = \sum_x f(x)p(x)$$

where x are all possible values that X can take.

Taking into account these two features, we can rewrite the expression for the current J, since $\mathbb{E}_\pi(l(\sigma)) = \sum_\sigma l(\sigma)\pi(\sigma)$ and so

$$J = \lim_{L\to\infty} \frac{\mathbb{E}_\pi\left[l(\sigma)\right]}{2L} = \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_\sigma l(\sigma)(1+w)^{l(\sigma)}}{W} =$$

$$= \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_\sigma l(\sigma)(1+w)^{l(\sigma)}}{\sum_\sigma w(\sigma)} =$$

$$= \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_\sigma l(\sigma)(1+w)^{l(\sigma)}}{\sum_\sigma (1+w)^{l(\sigma)}}$$

To get the final equation for the current J, we need a further step: we know that the TASEP is half-filled, which implies that the number of free particles $l(\sigma) = l$ goes from 0 to l and, for each value of l, there is a number $n(l)$ of configuration with that amount of particles which can move.

Then it is possible to state that:

$$J = \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_\sigma l(\sigma)(1+w)^{l(\sigma)}}{\sum_\sigma (1+w)^{l(\sigma)}} = \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_{l=1}^{L} l\ n(l)(1+w)^l}{\sum_{l=1}^{L} n(l)(1+w)^l}$$

Next, a formula for $n(l)$ is obtained as follows: we have to count the number of ways to divide L particles in l distinct groups and the number to divide L holes in l distinct groups. Then we count the number of configurations having $\sigma_1 = 1$ by fixing the number l_1, which represents the length of the first "particle-train" starting from σ_1, and then we multiply the number of configuration by l_1, due to the fact that the first set of particles has exactly l_1 ways to choose inside it the particle in first site. Note that the same reasoning can be applied for holes, so we multiply by a factor 2. Since L objects can be divided in l ordered groups in $\begin{pmatrix} L-1 \\ l-1 \end{pmatrix}$ ways,

$$n(l) = 2\sum_{l_1=1}^{L-l+1} l_1 \begin{pmatrix} L-l_1-1 \\ l-2 \end{pmatrix} \begin{pmatrix} L-1 \\ l-1 \end{pmatrix}.$$

If we put inside the previous equation,

$$J = \lim_{L\to\infty} \frac{1}{2L} \frac{\sum_{l=1}^{L} \sum_{l_1=1}^{L-l+1} l_1 \begin{pmatrix} L-l_1-1 \\ l-2 \end{pmatrix} \begin{pmatrix} L-1 \\ l-1 \end{pmatrix} l(1+w)^l}{\sum_{l=1}^{L} \sum_{l_1=1}^{L-l+1} l_1 \begin{pmatrix} L-l_1-1 \\ l-2 \end{pmatrix} \begin{pmatrix} L-1 \\ l-1 \end{pmatrix} (1+w)^l}$$

In order to get the final evaluation, let us write $l = \alpha L$, $l_1 = \alpha_1 L$ and use the leading order approximation

$$\begin{pmatrix} n \\ \alpha n \end{pmatrix} \approx e^{nI(\alpha)}$$

where $I(\alpha) = -\alpha ln\alpha - (1-\alpha)ln(1-\alpha)$. Since $\alpha = l/L$ and $\alpha_1 = l_1/L$, both α and α_1 are real numbers contained in the the interval $[0,1]$, the sum is replaced by integration from 0 to 1:

$$J = \lim_{L\to\infty} \frac{1}{2L} \frac{\int_0^1 d\alpha \int_0^1 d\alpha_1 \ \alpha \ \alpha_1 \ \exp[L((1-\alpha_1)I(\frac{\alpha}{1-\alpha_1}) + I(\alpha) + \alpha ln(1+w))]}{\int_0^1 d\alpha \int_0^1 d\alpha_1 \ \alpha_1 \ \exp[L((1-\alpha_1)I(\frac{\alpha}{1-\alpha_1}) + I(\alpha) + \alpha ln(1+w))]}.$$

Calling now $f(\alpha, \alpha_1) = (1-\alpha_1)I(\frac{\alpha}{1-\alpha_1}) + I(\alpha) + \alpha ln(1+w)$,

hence, by saddle-point method,

$$J = \lim_{L\to\infty} \left[\frac{1}{2}\overline{\alpha} + O\left(\frac{1}{L}\right) \right]$$

where $\overline{\alpha}$ is the value of a that maximizes $f(\alpha, \alpha_1)$.

$f(\alpha, \alpha_1)$ is a decreasing function of α_1, the choice $\alpha_1 = 0$ yields

$$\overline{\alpha} = \frac{\sqrt{1+w}}{1+\sqrt{1+w}} \text{ and consequently } J = \frac{1}{2}\frac{\sqrt{1+w}}{1+\sqrt{1+w}}$$

Finally we found the analytical result for the current in an half-filled TASEP. Just few considerations before moving to the next topic: for very small values of w, i.e. $w = O(\frac{1}{L})$, the model behave as the serial case and the current is exactly $\frac{1}{4}$; moreover J is an increasing function of w and $J \to \frac{1}{2}$ as $w \to \infty$.

1.5 Numerical results for the current

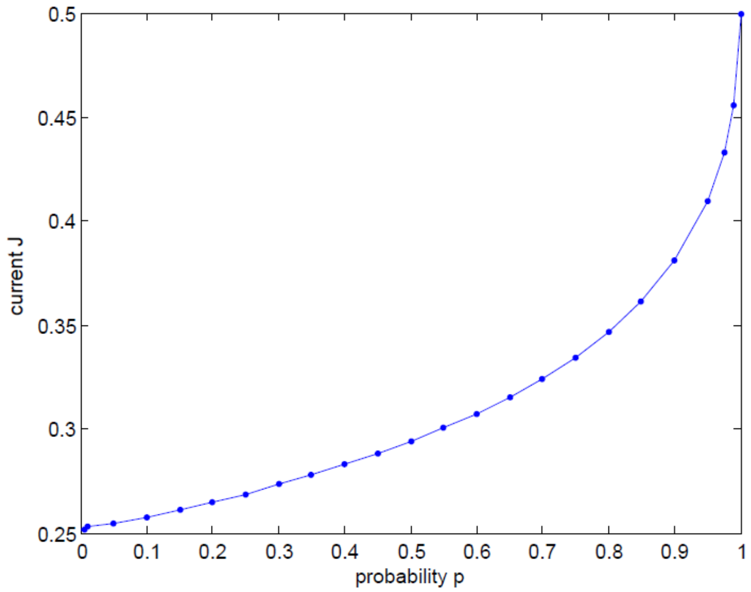


Figure 1.2: Graph representing the relationship between current J and probability p. This graph was obtained running simulations in a PCA (probabilistic cellular automata), which follows the dynamics of a half-filled parallel TASEP with a number $L = 100$ particles for L^3 iterations

Chapter 2

Blockage problem for the PCA-TASEP

In this chapter we will consider a very easy case of blockage problem for the parallel TASEP: we suppose, without loss of generality, to have a blockage between the site $2L$ and 1, such that, if p is the probability for each particle to jump to the next site, a particle in $\sigma_2 L$ will jump with probability $p(1-\varepsilon)$ into σ_1 if the latter there is a hole. Even though it may appear a very simple situation, it is shown by numerical results that, even if the blockage acts as local perturbation on the system, it has global effect on the whole model. Moreover, it is possible to find a mathematical relation between the value ε and the current J only when the TASEP is completely parallel, or equivalently when $w \to \infty$; this means that all free particles move with probability $p = 1$, except the one in state $2L$, that advances with $1-\varepsilon$ probability. In the following sections we will explain why it is required this situation to compute the current.

2.1 Transition probabilities in blockage problem

As we did in section 1.2, we first define the transition probabilities for the Markov chain: we define the weight of the transition from a configuration σ to a configuration

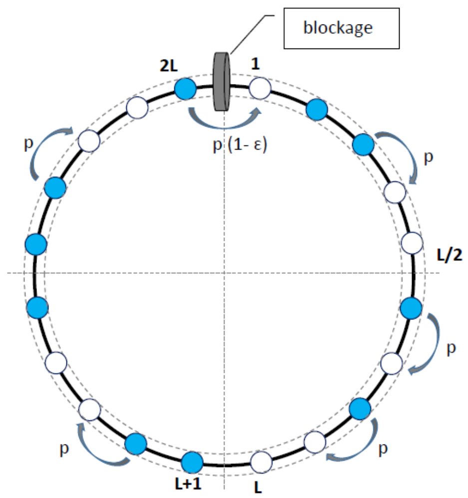


Figure 2.1: Representation of the model for PCA-TASEP with a blockage between sites 0 and $2L$

τ by

$$w(\sigma,\tau)=\begin{cases} w^n(1-\varepsilon\mathbb{1}_{\{\sigma_{2L}=1,\tau_{2L}=0\}}(\sigma)) & \text{if } \tau \text{ can be reached by moving } n \text{ particles in } \sigma \\ 0 & \text{otherwise} \end{cases} \tag{2.1}$$

where $0<\varepsilon<1$ and $\mathbb{1}_{\{\sigma_{2L}=1,\tau_{2L}=0\}}(\sigma)$ is the *indicator function* of the event $\{\sigma_{2L}=1,\sigma_1=0\}$ which assume value 1 only if the condition is satisfied, 0 otherwise. Hence the PCA-TASEP with one blockage has the following transition probabilities

$$P_\varepsilon(\sigma,\tau)=\frac{w(\sigma,\tau)}{w(\sigma)}.$$

It is crucial that now the model does not satisfy anymore the *global balance principle*; this causes the impossibility to find the exact form for the stationary distribution $\pi(\sigma)$, that helped us to find the current J in section 1.4. This implies that for finite values of w, the behaviour of the system can be seen only by running simulations and discussed just from a numerical point of view. However, for $w\to\infty$, where all particles actually move except the one in site $2L$, particular symmetry is preserved during each iteration.

2.2 The particle-hole symmetry

Definition 2.2.1. *A configuration σ is a* particle-hole configuration *if, for all $i = 1, ..., 2L$, the ith site $\sigma_i = 1 - \sigma_{2L-i+1}$. Moreover we denote with PH the set containing all the possible particle-hole configurations.*

We focus on the particle-hole symmetry, because for deterministic parallel TASEP, i.e. for $w \to \infty$, or equivalently for jump probability $p = 1$, it is preserved by the dynamics even in the presence of a blockage. This will give us a powerful tool for the computation of the current in this new problem. Let us establish some notation and denote the transition probability for the Markov chain as follows

$$P_{\varepsilon,\infty}(\sigma, \tau) = \lim_{w\to\infty} P_\varepsilon(\sigma, \tau).$$

We shall now describe and prove some very important properties of the particle-hole configurations:

Property 2.2.2. *For all configurations $\sigma \in PH$, if exists a configuration τ , which is reachable from σ ($P_{\varepsilon,\infty}(\sigma, \tau) > 0$), then $\tau \in PH$ for any $0 < \varepsilon < 1$.*

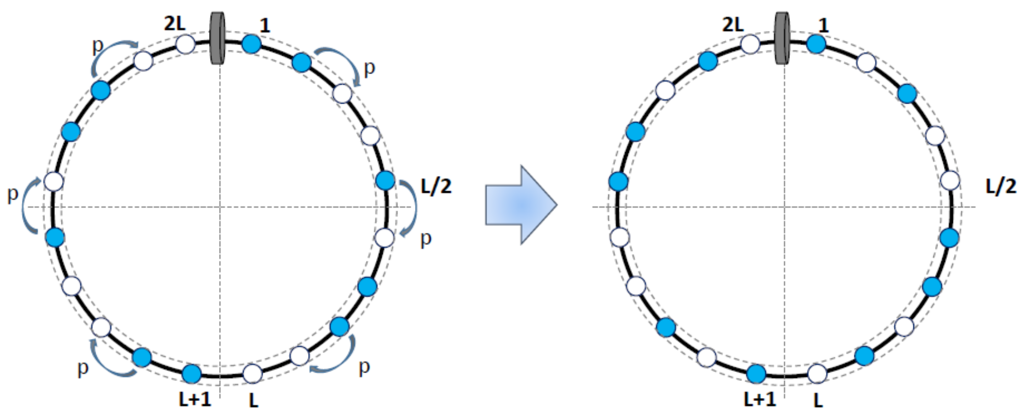


Figure 2.2: Example of particle-hole configuration and graphical explanation for property 2.2.2.

Proof. Since the configuration σ is symmetric, $\sigma_i = 1 - \sigma_{2L-i+1}$ and, moreover, $\sigma_{i+1} = 1-\sigma_{L-i}$ and $\sigma_{i-1} = 1-\sigma_{L-i+2}$. We can prove that after a step of our dynamics $\tau_i = 1 - \tau_{2Li+1}$ by considering all the possible local configurations $(\sigma_{i1}, \sigma_i, \sigma_{i+1})$.

Suppose for instance that initially in i there is a particle, $\sigma_i = 1$. If the particle is free to move, i.e. if $\sigma_{i+1} = 0$, due to the particle-hole symmetry we will have that in $L - i$ there is a particle free to move. Hence $\tau_i = 0$ and $\tau_{L-i} = 1$. If the particle in i is not free to move, i.e. if $\sigma_{i+1} = 1$, due to the particle-hole symmetry we will have holes both in holes both in $L - i$ and in $L - i + 1$. Hence $\tau_i = 1$ and $\tau_{L-1} = 0$.

The two remaining case can be treated analogously, considering the configuration σ in the sites $i - 1$ and $L - i + 1$. For the sites i and L the proof is similar □

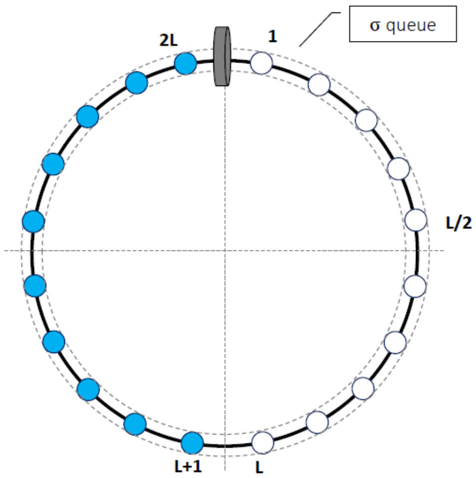
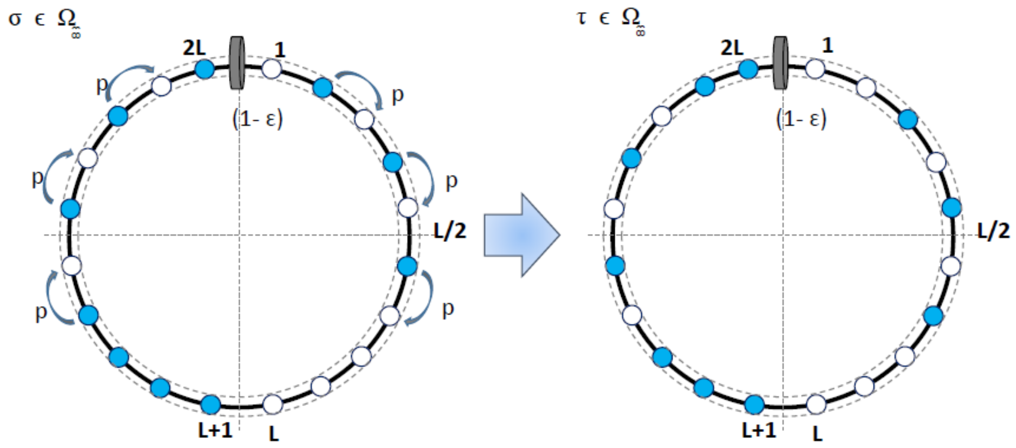


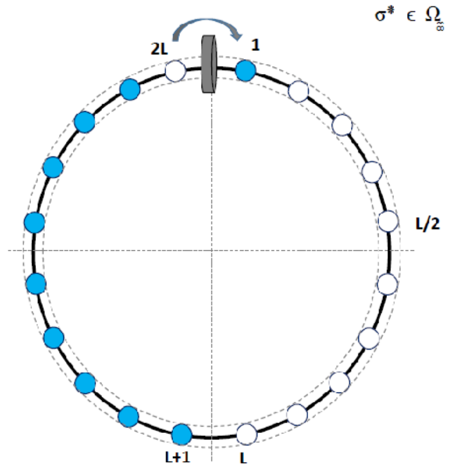
Figure 2.3: Representation for σ_{queue}

Property 2.2.3. *For the Markov chain defined by the transition probability $P_{\varepsilon,\infty}(\sigma, \tau)$, all the states which are not particle-hole symmetric ($\sigma \notin PH$) are transient.*

Proof. For this proof, we observe a particular configuration that we will call σ_{queue}: the particular feature of σ_{queue} is the fact that the first half of the sites $\{1, ..., 2L\}$ is empty, while the second one $\{L + 1, ..., 2L\}$ is filled. For all initial configuration, with a probability $p > \varepsilon^{2L}$ the chain reaches state σ_{queue} after $2L$ iterations. Note



(a) Transition from a configuration $\sigma \in \omega_\infty$ to a configuration τ which still belongs to Ω_∞. This figure helps to understand why all configurations $\in \Omega_\infty$ are recurrent states for the Markov chain



(b) The first configuration after σ_{queue} is a configuration $\in \Omega_\infty$

Figure 2.4: Graphical proof for property 2.2.3 applied to the set Ω_∞

that σ_{queue} is of course a *particle-hole configuration*. This shows that there is a finite probability to arrive in a symmetric state after $2L$ steps starting from a generic state σ, and hence with probability 1 we will never visit again σ due to property 2.2.2. □

In the previous property, we stated that all states not in PH are transient.

However a more accurate specification can be made about the transient states of the parallel TASEP with one blockage problem, and consequently we will be able to identify the recurrent states, necessary for specify the new stationary distribution.

We will call Ω_∞ the set of particular particle-hole configurations in which, in the first half of the ring $\{1, ..., L\}$, all particles are free to move.

When the chain arrives in one this configuration at some time, in the subsequent iterations all the particle in $\{1, ..., L\}$ will be free to move, due to the fact that all the particles, moving with probability 1 in $\{1, ..., L-1\}$, can never reach the preceding particle. Regarding the site L, if there is a particle in that site, we are sure that it is able to jump thanks to the symmetry equation $\sigma_i = 1 - \sigma_{2L-i+1}$:infact, computed for $i = L$, it yields to the result that $\sigma_{L+1} = 0$.

To prove that all the states outside Ω_∞ are transient, we argue as we did in the proof for property 2.2.3: once the model reaches σ_{queue}, infact, the first particle-hole configuration which follows is the case of $\sigma \in \Omega_\infty$ is the one with one free patircle in the first half; moreover, thanks to property 2.2.2, we know that the chain will have all configurations $\in \Omega_\infty$.

2.3 Stationary distribution in blockage problem

We identified the recurrent states of the Markov chain $P_{\varepsilon,\infty}(\sigma, \tau)$ as all the particle-hole configurations $\in \Omega_\infty$, therefore the stationary distribution $\pi_{\varepsilon,\infty}$ now is supported by Ω_∞ where the chain is manifestly ergodic. This means that is now possible to compute the stationary measure. Let us look at the site $2L$: it can be noticed that, if $\sigma_{2L} = 1$, the blockage is driven by a Bernoulli scheme success-insuccess, with success probability ε; this means that a binary variable can be associated to the blockage: for example we can treat this binary variable as a traffic light which can be "green" with probability $1 - \varepsilon$ and "red" with probability ε. If it is "green" we have, at the successive step, $\tau_{2L} = 0$, while we have $\tau_{2L} = 1$ otherwise.

Due to the symmetry, we can say that the probability of each state can be written in terms of red and green lights. In particular, when the particle has passed the blockage, and therefore $\sigma_{2L} = 0$ and $\sigma_1 = 1$, we know that, in the next iteration, we will obtain for sure $\tau_1 = 0$, $\tau_2 = 1$. By the symmetry property we can also state that $\tau_{2L} = 1$.

Hence the introduction of a new particle in the half-ring $\{1, ..., L\}$ depends only the number of red lights:

$$\pi_{\varepsilon,\infty}(r(\sigma)) = (1-\varepsilon)^{r(\sigma)}\varepsilon^{L-2r(\sigma)}$$

The exponent $L - r(\sigma)$ is due to the fact that each green light occupies the site of the particle and the subsequent one, which is for sure empty.

2.4 The current for half-filled TASEP with one blockage

We know, for definition 1.4.1, that the current $J_{\varepsilon,\infty} = \lim_{\Lambda\to\infty} \pi_{\varepsilon,\infty}(\sigma_i = 1, \sigma_{i+1} = 0)$, but, as we did in the section 1.4, we express the current in therms of the expectation of the number of free particle with respect to the stationary distribution $\pi_{\varepsilon,\infty}$. Let $r(\sigma)$ the number of free particles in the first half of the circle, i.e. sites $\{1, ..., L\}$, because in stationary conditions we have always particle-hole configurations, it's obvious that also in $\{L + 1, ..., 2L\}$ there are $r(\sigma)$ free particles,

$$J_{\varepsilon,\infty} = \lim_{L\to\infty} \frac{\mathbb{E}_{\pi_{\varepsilon,\infty}}[r(\sigma)]}{L}$$

where

$$\mathbb{E}_{\pi_{\varepsilon,\infty}}(r(\sigma)) = \sum_{\sigma} r(\sigma)\pi_{\varepsilon,\infty}(\sigma) = \sum_{\sigma} r(\sigma)(1-\varepsilon)^{r(\sigma)}\varepsilon^{L-2r(\sigma)}.$$

The number of free particles is $r(\sigma) = r$, with r that can assume values from 0 to $L/2$, and there are $n(r)$ particle-hole configurations with r free particles. So, again, we rewrite the sum substituting r as the index. Since $n(r)$ is the number of ways to divide $L - r$ holes in r groups, we get

$$\mathbb{E}_{\pi_{\varepsilon,\infty}}(r) = \sum_{r=0}^{L/2} r \begin{pmatrix} L-r \\ r \end{pmatrix} (1-\varepsilon)^r \varepsilon^{L-2r}.$$

Since we need to apply the *sample point method* as in the previous chapter, we divide $\mathbb{E}_{\pi_{\varepsilon,\infty}}(r)$ by 1 and express the denominator as

$$\sum_{r=0}^{L-r} \left(\begin{array}{c} L-r \\ r \end{array} \right) (1-\varepsilon)^r \varepsilon^{L-2r} = 1$$

by using the Newton Binomial Formula. So the equation becomes

$$\mathbb{E}_{\pi_{\varepsilon,\infty}}(r) = \frac{\sum_{r=0}^{L/2} r \left(\begin{array}{c} L-r \\ r \end{array} \right) (1-\varepsilon)^r \varepsilon^{L-2r}}{\sum_{r=0}^{L-r} \left(\begin{array}{c} L-r \\ r \end{array} \right) (1-\varepsilon)^r \varepsilon^{L-2r}} \approx \frac{\sum_{r=0}^{L/2} r \left(\begin{array}{c} L-r \\ r \end{array} \right) (1-\varepsilon)^r \varepsilon^{L-2r}}{\sum_{r=0}^{L/2} \left(\begin{array}{c} L-r \\ r \end{array} \right) (1-\varepsilon)^r \varepsilon^{L-2r}}$$

Since $L \to \infty$.

We are almost there, we need just few substitutions more: we define the number $x = \frac{r}{L}$; hence the number $x \in \mathbb{R}$ and $0 \leq x \leq \frac{1}{2}$. Putting x in the previous formula, we can now use the *leading therm approximation* for the binomial coefficient and replace the sum with the integral $\int_0^{1/2} dx$, so that we can apply the saddle method point.

$$\mathbb{E}_{\pi_{\varepsilon,\infty}}(r) \approx L \frac{\int_{x=0}^{1/2} x \left(\begin{array}{c} (1-x)L \\ xL \end{array} \right) (1-\varepsilon)^{xL} \varepsilon^{(1-2x)L} dx}{\int_{x=0}^{1/2} \left(\begin{array}{c} (1-x)L \\ xL \end{array} \right) (1-\varepsilon)^{xL} \varepsilon^{(1-2x)L} dx}$$

So the final equation for the current $J_{\varepsilon,\infty}$ is

$$J_{\varepsilon,\infty} = \lim_{L\to\infty} \frac{\int_0^{1/2} x\exp[L((1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x})]dx}{\int_0^{1/2} \exp[L((1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x})]dx}$$

$$J_{\varepsilon,\infty} \approx \overline{x}$$

where $\overline{x}$ is the value that maximizes

$$f(x) = (1-2x)\ln\varepsilon + x\ln(1-\varepsilon) - x\ln(1-\varepsilon) - x\ln\frac{x}{1-x} - (1-2x)\ln\frac{1-2x}{1-x}$$

and by derivation it is possible to prove that

$$J_{\varepsilon,\infty} = \frac{1-\varepsilon}{2-\varepsilon}$$

. The computation proves that, in completely parallel context, a very small perturbation of the transition probabilities in a single site extends its effect all over the volume without any fading.

2.5 Results of simulations

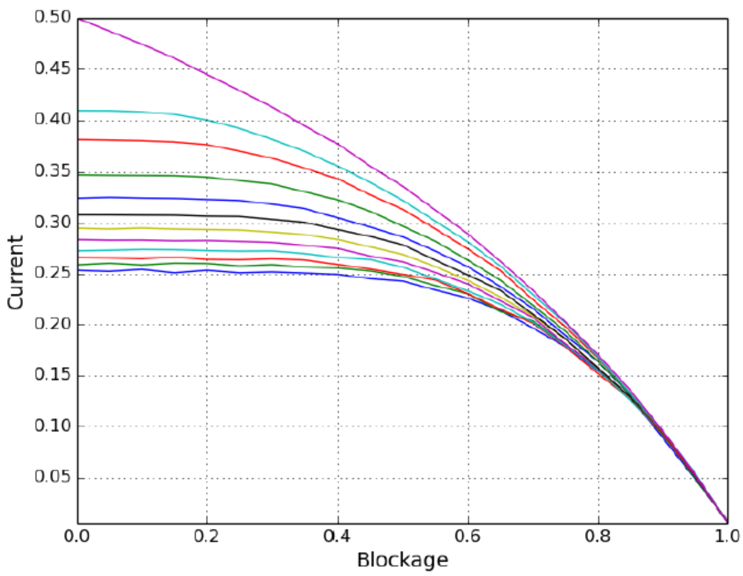


Figure 2.5: Profile curves of the PCA-TASEP current for different values of the probability p, from $p = 1$ to $p = 1/L$

It clearly appears that except in the case $p = 1$, where the current decrease with a finite slope for all $\varepsilon > 0$, the decrease of J starts only after a certain value of the blockage. For all the probabilities except $p = 1$, we note that until some value of the blockage ε, the current tends to remain constant, especially for very low values of p.

Chapter 3

Variable speed PCA-TASEP

In this chapter, we will analyze the effect of the possibility for the particles to have a variable speed on the current of the system. We introduce a very simple concept of variable speed: we give, to each free particle, the chance to make more jumps when it is possible; in our case, the favorable condition is that, for any $i \in \{1, ..., 2L\}$, $\sigma_i = 1$ and $\sigma_{i+1} = 0$ and $\sigma_{i+2} = 0$, which means that we are studying the case with two possible jumps.

In the physical world, this can be interpreted like a car in a traffic jam that, when a stretch of the road is empty, moves forward faster and further: infact, in the discretized model, the acceleration corresponds to the possibility to jump twice during the same iteration of the system. The second jump is driven by a probability q: when $q = 0$ we come back to the situation studied in chapter 1, while, for $q = 1$, the particle will jump whenever the situation allows it. In these circumstances we cannot use the definition of transition probability of chapter 1 and we do not know if the *global balance principle* still holds, so we are not able to describe the Markov chain of the PCA-TASEP under variable speed condition. Anyway, we still are able to run simulations of the system's evolution to observe how different values of probability q affect the current $J_{p,q}$: infact we can measure the value of $J_{p,q}$ simply by counting the average number of free particles during the evolution of the system and weighting such value with the total volume $2L$ of the lattice. This value is close enough to the real one according to the *Law of Large Numbers*, which states that the average of the results obtained from a large number of trials should be close to the expected value, and will tend to become closer as more trials are performed.

3.1 Numerical results for the current $J_{p,q}$

In this section we provide the results, reported using graphs, for the current $J_{p,q}$: the number of the particles in the lattice is 100; consequently the volume is 200. We choose $L^3 = 1000000$ as the number of iterations, so that the error on the results is of order 10^{-3}.

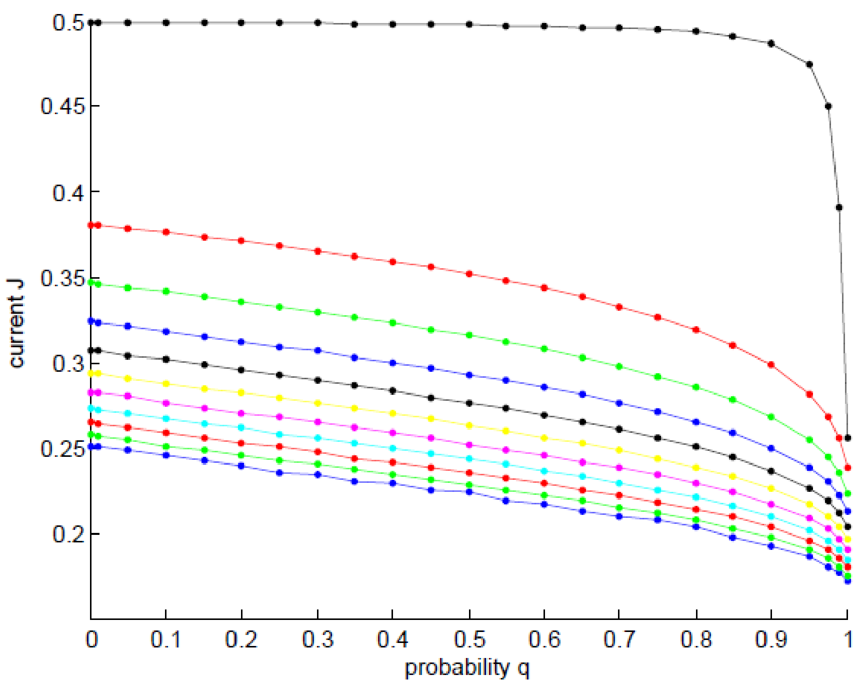


Figure 3.1: Graph of the simulation of the PCA considering the case of variable speed

We used as values of p in the first figure $\{0.01, 0.1, 0.2, 0.3, 0.4, 0.5, 0.6, 0.7, 0.8, 0.9, 1\}$. The main result we appreciate is the fact that, as q increases, the current tends to decrease: for small values of p, it can be seen that it decreases almost linearly until q reaches the value 0.9, while for the completely parallel case, the current remains almost constant, but again, for $q = 0.9$, it plunges immediately. Then we can state that 0.9 it is a value, for all the different p such that the speed of falling of the current becomes considerably higher, and this behaviour becomes more appreciable for higher values of the probability p. This can be explained physically: let us denote as "train" a group of adjacent particles. It is easy to prove that an higher number of trains implies a lower value for $J_{p,q}$. Hence q drives also the probability of the formations of trains inside the lattice: for small values of p most likely a particle will remain in its current state without jumping, but if it jumps, for high values of q, it is almost certain that it will approach the next particles by jumping further and, after few iterations, there will be a train. This effect is limited as the model tends to be completely parallel, however for $q > 0.9$, all the particles will make the second transition and they will form trains.

This second figure shows the behaviour of the current for higher values of p, and it confirms the previous qualitative analysis we made and depicts better how the rate of decrease after $q = 0.9$ is higher for $p = \{0.9, 0.975, 0.99, 1\}$.

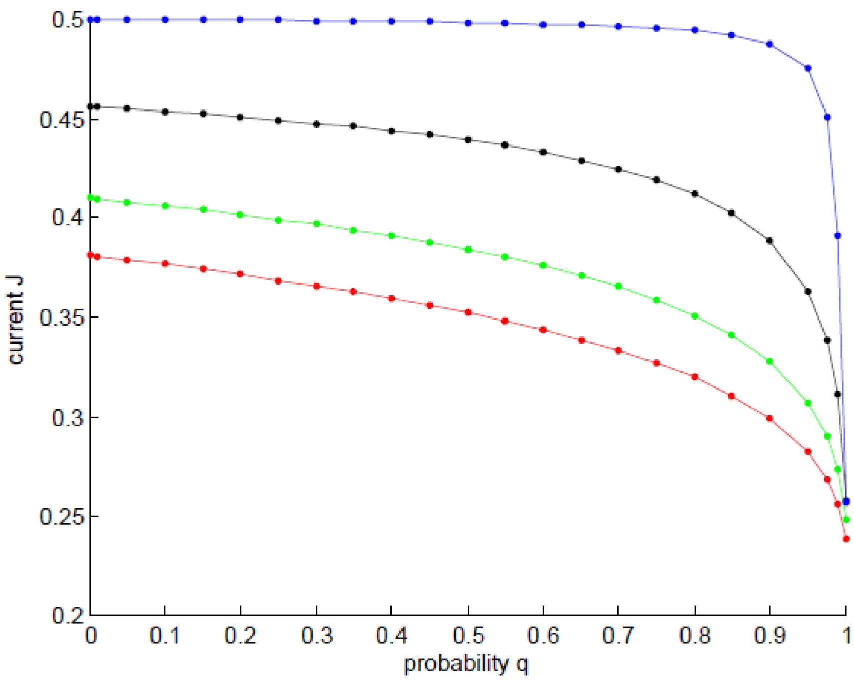


Figure 3.2: Graph for high values of p, i.e. $\{0.9, 0.95, 0.99, 1\}$

3.2 The blockage problem under variable speed condition

After the description of this new model, we now focus on the effect of a simple blockage on the system as we did in chapter 2: the blockage is again defined as the decrease of the probability of moving from σ_{2L} to σ_1, from the value p to $p(1-\varepsilon)$, with $0 \leq \varepsilon \leq 1$. Differently from before, we are not sure if the particle-hole symmetry still holds, so we can just have numerical results without a proper mathematical explanation. We will provide some graphs reporting the outcomes of our experiments.

In the experiments we used the values $p \in \{0.75, 0.8, 0.85, 0.9, 0.925, 0.95, 0.975, 1\}$ and $\varepsilon \in \{0, 0.2, 0.3, 0.5\}$. From the figures, we note that, for low values of probability q, the value of the current J tends to remain constant or at least to diminish by a very small amount. This can be explained by the fact that the system is in a situation very close to the one analyzed in section 2.4, since the effects of the probability q are

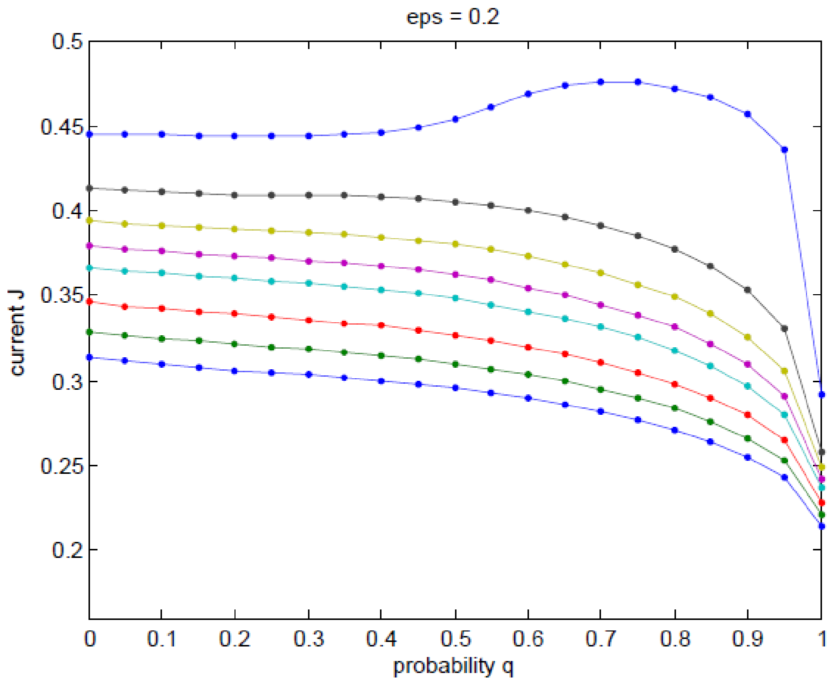


Figure 3.3: Graph of the current J in variable speed situation with a blockage ($\varepsilon = 0.2$) between sites 0 and $2L$

negligible and are nullified by the blockage. This happens for all the three values of ε we tested.

An unexpected phenomenon is the presence of a region were J actually increases despite the blockage for very high values of probability p. We said in the previous section, that the main effect of the probability q on the system is the formation of particles' trains in the lattice, that determine a quick decay for the current. Moreover the blockage causes an higher density of particles in the second half of the ring $\{L+1, ..., 2L\}$ and the higher are the values of ε, the higher is the density. However the blockage has the property to prevent the formation of trains and it distributes more uniformly the particles from the second half into the first half of the ring. It must be taken into account also the effect of q: if a particle is able to pass the block, then it can set a gap from the preceding particle thanks to the high probability to jump twice, and this actually generates an increase of the number of free particles, and, consequently, of the current J.

However, even if we said that in this situation the number of possible trains is considerably reduced in the first half of the lattice, this benefit gets lost as $q > 0.8$, because the particles approach faster to the right side of the blockage, where the density is higher. Hence very long trains are generated and this is the reason for the decay of the value of the current J.

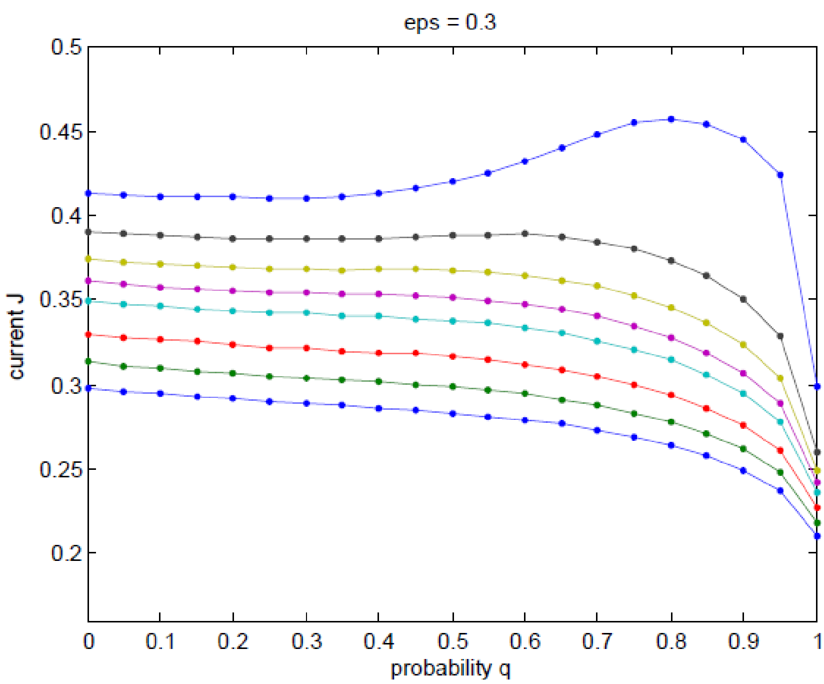


Figure 3.4: Graph of the current J in variable speed situation with a blockage ($\varepsilon = 0.3$) between sites 0 and $2L$

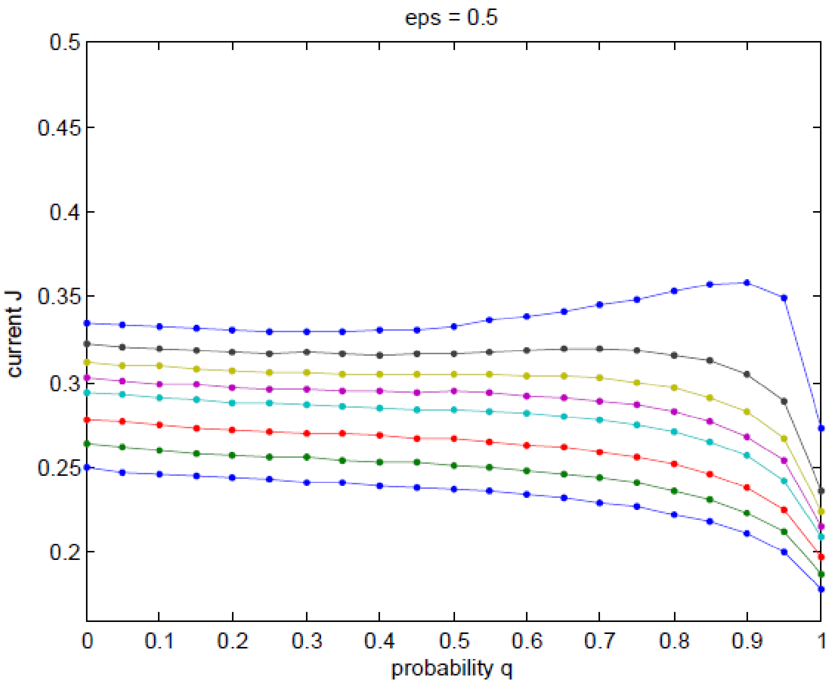


Figure 3.5: Graph of the current J in variable speed situation with a blockage ($\varepsilon = 0.5$) between sites 0 and $2L$

Chapter 4

Appendix

4.1 Rule-184 Cellular Automata

Rule 184 is a one-dimensional binary cellular automaton rule, notable for solving the majority problem as well as for its ability to simultaneously describe several, seemingly quite different, particle systems:

Rule 184 can be used as a simple model for traffic flow in a single lane of a highway, and forms the basis for many cellular automaton models of traffic flow with greater sophistication. In this model, particles (representing vehicles) move in a single direction, stopping and starting depending on the cars in front of them. The number of particles remains unchanged throughout the simulation. Because of this application, Rule 184 is sometimes called the "traffic rule".

In each step of its evolution, the Rule 184 automaton applies the following rule to determine the new state of each cell, in a one-dimensional array of cells:

current pattern	111	110	101	100	011	010	001	000
new state	1	0	1	1	1	0	0	0

The rule set for Rule 184 may also be described intuitively, in several ways:

At each step, whenever there exists in the current state a 1 immediately followed by a 0, these two symbols swap places.

At each step, if a cell with value 1 has a cell with value 0 immediately to its right, the 1 moves rightwards leaving a 0 behind. A 1 with another 1 to its right remains in place, while a 0 that does not have a 1 to its left stays a 0 (traffic flow modeling)

4.2 Implemented codes

The function *Get_Model* gets as input the number of desired particles living in the TASEP and returns a list v, whose entries are l ones and l zeros (*half-filled case*).Initially the algorithm initialize an empty list, which gets filled by $2l$ holes using a for cycle. Then a vector *ones* containing the indexes for the particles is created and with another for cycle the particles are inserted into the list.

The procedure *Get_Current* has as inputs the half-filled list v, the number of iterations n, the two values for probabilities p and q and the blockage ε, and it returns the value of the current.

In order to have a parallel behaviour, we use an auxiliary array *ind*, where we store the indexes of the free particles, identified by scanning the list v and checking if, for each site j, the condition $v[j] = 1$ and $v[j+1] = 0$ is verified. Note that the array *ind* is modified at each iteration of the outer for cycle.

After defining *ind*, we update the list v with the second for cycle: using a uniform random numbers generator, which produces numbers between 0 and 1, and comparing the outcome with the value p is possible to simulate the randomness of the PCA-TASEP. A second comparison is made with q, in the case of variable speed PCA-TASEP. If the random number is higher than p (or q), then the particle jumps, otherwise it remains in its current site. At each iteration, we add the number of free particles to the variable *count* and, after the outer for cycle, we get the estimation for the current simply by dividing *count* by the number of iterations and by the dimension of the lattice

```python
'''
Created on 01/ott/2015

@author: Federico
'''
def Get_Model(l):
    from random import sample
    v=[];
    ones = sample(xrange((2*l)- 1), l);
    for i in range(2*l):
        v.append(0);
    for i in ones:
        v[i] = 1;
    return v


def Get_Current(v, n, p, q, eps):
    from random import random
    count=0;
    for i in range(n):
        ind =[];
        for j in range(0,len(v)):
            if j!=len(v)-1 and v[j]==1 and v[j+1]==0:
                ind.append(j);
            elif j==len(v)-1 and v[j]==1 and v[0]==0:
                ind.append(j);


        for k in ind:
            if random() <= p:
                if k < len(v)-2:
                    v[k]=0;
                    v[k+1]=1;
                    if random() <= q and v[k+2]==0:
                        v[k+1] = 0;
                        v[k+2] = 1;
                elif k == len(v)-2:
                    v[k] = 0;
                    v[k+1] = 1;
                    if random() <= q*(1-eps) and v[0]==0:
                        v[k+1] = 0;
                        v[0] = 1;
                elif random() <= p*(1-eps):
                    v[k]=0;
                    v[0]=1;
                    if random() <= q and v[1]==0:
                        v[0]=0;
                        v[1]=1;

       # print "at iteration ", i;
        #print v;
        count = count + len(ind);

    current = count/ float(n*len(v));
    return current
```

Figure 4.1: Codes implemented in python language

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